

Phase shift operator and cyclic evolution in finite dimensional Hilbert space

Ramandeep S. Johal*

Department of Physics, Panjab University,
Chandigarh -160 014, India.

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We address the problem of phase shift operator acting as time evolution operator in Pegg-Barnett formalism. It is argued that standard shift operator is inconsistent with the behaviour of the state vector under cyclic evolution. We consider a generally deformed oscillator algebra at q root of unity, as it yields the same Pegg-Barnett phase operator and show that shift operator within this algebra meets our requirement.

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In recent years, Pegg-Barnett (PB) formalism has attracted wide attention as a theory for quantum phase [1], [2]. Alongside, the subject of quantum algebras and their realizations in terms of q -deformed oscillators has also been studied with great interest [3]- [5]. The problem of quantum phase has also been pursued from q -deformation theoretic point of view. There are certain justifications for this approach. A feature of a q -deformed theory or framework is that one can identify an inherent scale in it, of magnitude $\sim |1 - q|$, where q is called deformation parameter. As $q \rightarrow 1$, one retrieves the undeformed or "classical" theory. Now PB formalism can also be looked upon as inherently q -deformed [6] in the above-said sense and in this case $q = \exp(i2\pi/s + 1)$, where $(s + 1)$ is the dimension of the Hilbert space. Secondly, the phase observable which is hermitian phase operator in PB theory, can be consistently defined only in a finite dimensional Hilbert space (FDHS). These two features form the motivation to study phase using a q -oscillator with $q = \exp(i2\pi/(s + 1))$ [7]. Firstly, q being root of unity, naturally truncates the q -oscillator to a FDHS. Secondly, infinite dimensional limit ($s \rightarrow \infty$) also corresponds to the deformation free ($q \rightarrow 1$) limit. However, the problem of negative norm in this representation was recognised later and there now exist representations of q -oscillator [8] or generally deformed oscillator [9] with positive norm for q as root of unity and for which the Pegg-Barnett phase operator can be consistently defined.

Let us first recapitulate relevant key points of PB formalism. Here the phase operator Φ and the number operator N are not canonically conjugate, but satisfy a complicated commutator

$$[\Phi, N] = \frac{2\pi\hbar}{s+1} \sum_{n,n'=-l}^l \frac{(n' - n)|n'\rangle\langle n|}{\exp[2\pi i(n - n')/(s + 1)] - 1}. \quad (1)$$

The eigenstates of Φ which form an orthonormal set of phase states, are related to the number states by Fourier transform

$$|\theta_m\rangle = \frac{1}{\sqrt{s+1}} \sum_{n=0}^s \exp(in\theta_m)|n\rangle, \quad (2)$$

where $\theta_m = \theta_0 + \frac{2\pi m}{s+1}$, $\{m = 0, 1, 2, \dots, s\}$. θ_0 is the arbitrary phase window which defines the phase angle interval 2π modulo, $\theta_0 \leq \theta_m < \theta_0 + 2\pi$. Apart from the hermitian phase operator Φ , the unitary phase operator $e^{i\Phi}$ is also of significance in PB theory. It acts as shift operator on number states

$$e^{i\Phi}|n\rangle = |n - 1\rangle, \quad n \neq 0 \quad (3)$$

$$e^{i\Phi}|0\rangle = e^{i(s+1)\theta_0}|s\rangle. \quad (4)$$

Thus the action of $e^{i\Phi}$ is cyclic and it steps down the number states by unity. Its adjoint acts as step up operator. Thus one can write a realization of unitary phase operator as

$$e^{i\Phi} = |0\rangle\langle 1| + |1\rangle\langle 2| + \dots + |s - 1\rangle\langle s| + e^{i(s+1)\theta_0}|s\rangle\langle 0|. \quad (5)$$

The operator dual to $e^{i\Phi}$ is the operator q^{-N} , which acts as shift operator on the phase states

$$q^{-N}|\theta_m\rangle = |\theta_{m-1}\rangle, \quad m \neq 0 \quad (6)$$

$$q^{-N}|\theta_0\rangle = |\theta_s\rangle. \quad (7)$$

Note that the apparent duality between the two kinds of shift operators seems incomplete due to the extra phase factor in Eq. (4) or the lack of corresponding factor in Eq. (7). This is due to the arbitrariness in the choice of phase window in PB formalism, while there is no such choice in the ground state eigenvalue of number operator, which is necessarily zero. Thus the realization of q^{-N} in terms of phase states is

$$q^{-N} = |\theta_0\rangle\langle \theta_1| + |\theta_1\rangle\langle \theta_2| + \dots + |\theta_{s-1}\rangle\langle \theta_s| + |\theta_s\rangle\langle \theta_0|. \quad (8)$$

Now the unitary phase shift operator q^{-N} can be thought as time evolution operator, which operated once on the phase state advances the phase by $2\pi/(s + 1)$. Thus if

*e-mail: raman%phys@puniv.chd.nic.in

we operate it $(s+1)$ times on a phase state, we complete one cycle and return to the same phase state. On the other hand, we have the results of Ref. [10], where it was shown that for cyclic evolution of harmonic oscillator in a general state $\sum_n c_n |n\rangle$, in FDHS, the state vector can change sign which depends on the dimensionality of the space; if $(s+1)$ is even sign changes, otherwise not. However, if we take q^{-N} as equivalent to time evolution operator, we note that according to realization of Eq. (8), the state vector always returns exactly to initial state, irrespective of the dimensionality of the space.

The purpose of this paper is to make the action of phase shift operator consistent with that of time evolution operator in the context of cyclic evolution in FDHS. We take as our model the recently proposed generally deformed oscillator [9], which has certain advantages over other approaches from algebraic point of view, namely, i) The creation and annihilation operators in PB theory do not form a closed algebra by themselves, and they do not go over to corresponding relations in the $s \rightarrow \infty$ limit ii) we can algebraically define PB phase operator in the approach of [9], iii) for q as root of unity, positive norm is also assured.

Briefly, in the approach of [9], new creation and annihilation operators are defined

$$A^\dagger = \sqrt{\mathcal{F}(q^N)} e^{-i\Phi}, \quad A = e^{i\Phi} \sqrt{\mathcal{F}(q^N)}, \quad q^N = q^{N+\eta}. \quad (9)$$

The action of these operators on generalized number states is

$$A^\dagger |n+\eta\rangle = \sqrt{\mathcal{F}(q^{n+\eta+1})} |n+\eta+1\rangle, \quad n \neq s \quad (10)$$

$$A^\dagger |s+\eta\rangle = e^{-i(s+1)\theta_0} \sqrt{\mathcal{F}(q^n)} |\eta\rangle \quad (11)$$

$$A |n+\eta\rangle = \sqrt{\mathcal{F}(q^{n+\eta})} |n+\eta-1\rangle, \quad n \neq 0 \quad (12)$$

$$A |\eta\rangle = \sqrt{\mathcal{F}(q^\eta)} e^{i(s+1)\theta_0} |s+\eta\rangle \quad (13)$$

$$q^N |n+\eta\rangle = q^{n+\eta} |n+\eta\rangle. \quad (14)$$

The parameter η is chosen such that i) the above defines a cyclic representation, ii) the function \mathcal{F} is hermitian and non-negative, iii) in $s \rightarrow \infty$ limit, A^\dagger and A go over to the creation and annihilation operators of the ordinary oscillator. Also the condition for cyclic representation ($\mathcal{F}(q^n) \neq 0$) in Eqs. (11) and (13) also ensures that one can consistently define unitary phase operator by inverting A^\dagger and A in Eq. (9). Note that this approach exactly recovers the PB phase operator.

However a significant fact that was missed in [9] is that in the above representation, $q^{\pm N}$ can also act as phase shift operator on the phase states. As one can easily see, its action gives $q^{-N} |\theta_m\rangle = q^{-\eta} |\theta_{m-1}\rangle$ and $q^{-N} |\theta_0\rangle = q^{-\eta} |\theta_s\rangle$, which is just same as Eqs. (6) and (7). However, as we show below, the significance of this operator lies in its being consistent with the results of cyclic evolution in FDHS [10]. As a solution for restoring the duality in $e^{i\Phi}$ and q^{-N} , we propose to modify the Eq. (2) as follows:

$$|\theta_m\rangle = \frac{1}{\sqrt{s+1}} \sum_{n=0}^s \exp(i(n+\eta)\theta_m) |n+\eta\rangle, \quad (15)$$

so that now we have

$$q^{-N} |\theta_m\rangle = |\theta_{m-1}\rangle, \quad m \neq 0 \quad (16)$$

$$q^{-N} |\theta_0\rangle = e^{-i2\pi\eta} |\theta_s\rangle. \quad (17)$$

The action of $e^{i\Phi}$ on the phase states remains as such, i.e. $e^{i\Phi} |\theta_m\rangle = \theta_m |\theta_m\rangle$. Moreover, the action of $e^{i\Phi}$ on the (new) number states remains same as before. Thus from Eq. (15)

$$|n+\eta\rangle = \frac{1}{\sqrt{s+1}} \sum_{m=0}^s \exp(-i(n+\eta)\theta_m) |\theta_m\rangle, \quad (18)$$

we can write

$$e^{i\Phi} |n+\eta\rangle = |n+\eta-1\rangle, \quad n \neq 0 \quad (19)$$

$$e^{i\Phi} |\eta\rangle = e^{i(s+1)\theta_0} |s+\eta\rangle. \quad (20)$$

Thus the duality between $e^{i\Phi}$ and q^{-N} is exactly obeyed, so that parameter η plays the role equivalent to θ_0 . We can as well write the following realization for modified unitary operator

$$q^{-N} = |\theta_0\rangle\langle\theta_1| + |\theta_1\rangle\langle\theta_2| + \cdots + |\theta_{s-1}\rangle\langle\theta_s| + e^{-i2\pi\eta} |\theta_s\rangle\langle\theta_0|. \quad (21)$$

Therefore, operating the above unitary operator $(s+1)$ times, we get

$$(q^{-N})^{s+1} |\theta_m\rangle = e^{-i2\pi\eta} |\theta_m\rangle. \quad (22)$$

Next, we are interested to know if under such cyclic evolution, the state vector changes sign or not. Thus if η is an integer, no change in sign occurs, while for η as half-odd integer, there is change in sign. Now the usual time evolution operator is $e^{-iHt/\hbar}$, where for the case of harmonic oscillator in FDHS [10], the hamiltonian H has the following energy spectrum

$$E_n = \hbar\omega \left(n + \frac{1}{2} + \frac{(s+1)}{2} \delta_{n,s} \right). \quad (23)$$

Thus under evolution through one time period, $t = 2\pi/\omega$, the state vector $|n\rangle$ is multiplied by the phase factor $\exp(-i2\pi\{n+1/2+(s+1)\delta_{n,s}/2\})$. On the other hand, if we consider time evolution through unitary shift operator q^{-N} , this means that state vector is multiplied by the factor $\exp(-i2\pi\{n+\eta\})$. Thus we note that for a harmonic oscillator in FDHS, for $n \neq s$, we have $\eta = 1/2$, whereas for $n = s$, $\eta = 1/2 + (s+1)/2$. So $(s+1)$ as even number is equivalent to η as half-odd integer, which from previous discussion, implies change in sign under cyclic evolution, whereas $(s+1)$ odd is equivalent to η as integer and consequently no change in sign of the state vector under one cycle. Also, the case

of infinite dimensional harmonic oscillator requires that $E_n = (n + 1/2)\hbar\omega$, which is consistent with $\eta = 1/2$.

Finally, it is interesting to note that states $|n + \eta\rangle$ can be obtained from usual number states $|n\rangle$, by applying a continuous unitary transformation i.e. when η is not an integer ($e^{-i\eta\Phi}|n\rangle = |n + \eta\rangle$). As was pointed out in [1], such continuous unitary transformations are useful to construct the phase-moment generating functions.

Concluding, we have argued that phase shift operator in standard PB formalism is inconsistent with cyclic evolution of harmonic oscillator in finite dimensional Hilbert space. To treat this, we have shown that phase shift operator of a generally deformed oscillator algebra at q root of unity, and which yields the same PB phase operator, can simulate the behaviour of time evolution operator for cyclic evolution. This also restores the duality in the actions of phase- and number- shift operators.

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